Computing Real Logarithm of a Real Matrix

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Abstract

In this paper we will be interested in characterizing and computing for a nonsingular real matrix $A \in \mathbb{R}^{n \times n}$ a real matrix $X \in \mathbb{R}^{n \times n}$ that satisfies $e^X = A$, that is, a logarithm of $A$. Firstly, we investigate the conditions under which such logarithm exists, unique, polynomial in $A$, or belongs to a particular class of matrices. Secondly, real Schur decomposition will be used to compute $X$.

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1 Introduction

Logarithms of matrices arise in various contexts. For example [1, 3, 8], for a physical system governed by a linear differential equation of the form

$$\frac{dy}{dt} = Xy,$$

where $X$ is n-by-n unknown matrix. From observations of the state vector $y(t)$, if $y(0) = y_0$ then we know that

$$y(t) = e^{tX}y_0.$$

By taking $n$ observations at $t = 1$ for $n$ initial states consisting of the columns of the identity matrix, we obtain the matrix $A = e^X$. Under certain conditions on $A$, we can then solve for $X$, that is $X = \log A$. This raises the question of how to compute a logarithm of a matrix. We show that $S(A)$, the solution set of this matrix equation, is nonempty if and only if $A$ is nonsingular.

In this paper we concerned with the real solvability of the matrix equation $e^X = A$ in case of real matrix $A$. Not every nonsingular real matrix have a real logarithm as the following example illustrates.

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Example 1 Let \( A = \text{diag}(1, -1) \). Then any logarithm of \( A \) is given by
\[
X = U \text{diag}(\log(1), \log(-1))U^{-1} = U \text{diag}(2\pi ij_1, i\pi(2j_2 + 1))U^{-1}
\]
where \( j_1, j_2 \in \mathbb{Z} \) and \( U \) is any nonsingular matrix commuting with \( A \). All these logarithms are matrices with noncomplex conjugate eigenvalues. Hence it can not be similar to a real matrix and no real logarithm of \( A \) can be obtained.

The existence of a real logarithm of a real matrix is discussed in Section 2. In Section 2, we also characterize such logarithm, that is, set the conditions for which \( X \) is polynomial in \( A \), symmetric, positive definite, or orthogonal.

The computation of a real logarithm \( X \) of a real matrix \( A \) arises in many system identification, one of which is the mathematical modeling of dynamic systems \([4]\). In Section 3, we propose a technique to compute such \( X \) based on the real Schur decomposition.

2 Characterization of a real logarithm

In the following theorem we give a set of conditions on the matrix \( A \) that guarantees the existence of a real logarithm of \( A \). We start by a lemma which constructs a real logarithm of a particular 2-by-2 matrices.

Lemma 2 The 2-by-2 real matrices of the form
\[
A_1 = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \quad \lambda > 0 \quad \text{and} \quad A_2 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad b \neq 0
\]
have real logarithms given by
\[
X_1 = \begin{bmatrix} \log\lambda & \pi \\ -\pi & \log\lambda \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} \theta & \mu \\ -\mu & \theta \end{bmatrix},
\]
respectively, where \( e^{\theta \pm i\mu} = a \pm ib \).

Proof. To prove this lemma, it is enough to show that \( e^{X_1} = A_1 \) and \( e^{X_2} = A_2 \). For the matrix \( X_1 \), there exists a nonsingular matrix \( V \) such that
\[
e^{X_1} = e^{V \text{diag}(\log\lambda + i\pi, \log\lambda - i\pi)V^{-1}} = V(-\lambda I)V^{-1} = -\lambda I = A_1.
\]
For \( X_2 \), it is clear that \( X_2 \) is a normal matrix, and since \( A_2 \) and \( X_2 \) are diagonalizable and commuting, they are simultaneously diagonalizable \([7]\). Then
\[
e^{X_2} = e^{U \text{diag}(\theta + i\mu, \theta - i\mu)U^{-1}} = U \text{diag}(a + ib, a - ib)U^{-1} = A_2.
\]
Next we set up a sufficient and necessary condition for the existence of a real logarithm $X$ of a real matrix $A \in \mathbb{R}^{n \times n}$. A proof of the following theorem can be found in Culver [5], and Ulig [6], however the proof we provide is different, it is based on the usage of real Jordan canonical form and the previous lemma.

**Theorem 3** Let $A \in \mathbb{R}^{n \times n}$ be a real matrix. Then there exists a real logarithm $X$ of a real matrix $A$ if and only if $A$ is nonsingular and each Jordan block of $A$ belonging to negative eigenvalue occurs an even number of times.

**Proof.** Let $X$ be a real logarithm of $A$, that is, $e^X = A$. By using [7, Th 3.4.5] each complex Jordan block (if exists) of any size occur in the Jordan canonical form of a real matrix in conjugate pairs. Hence we may suppose that the Jordan canonical form of $X$ is

$$J_X = \text{diag}(J_{m_1}(x_1), \ldots, J_{m_r}(x_r), B_{2m_{r+1}}, \ldots, B_{2m_p}),$$

where $x_1, \ldots, x_r$ are real, $x_{r+1}, \ldots, x_p$ are complex, and $x_1, x_2, \ldots, x_p$ are not necessarily distinct and $B_{2m_s} = \text{diag}(J_{m_s}(x_s), J_{m_s}(\overline{x_s}))$. The Jordan canonical form $J_A$ of $A = e^X$ has the form

$$J_A = \text{diag}(J_{m_1}(e^{x_1}), \ldots, J_{m_r}(e^{x_r}), B_{2m_{r+1}}, \ldots, B_{2m_p}),$$

where $B_{2m_s}' = \text{diag}(J_{m_s}(e^{x_s}), J_{m_s}(e^{\overline{x_s}}))$, and $\overline{x}$ denotes the complex conjugate of $x$. Clearly, $e^{x_k} \neq 0$ for any $x_k \in C$, then $A$ must be nonsingular. Moreover, $e^{x_k} < 0$ only if $\text{Im}(x_k) \neq 0$, in which case $e^{x_k} = e^{\overline{x_k}}$. Thus negative eigenvalues of $A$ must be associated with Jordan blocks which occur in pairs.

Conversely, let $A \in \mathbb{R}^{n \times n}$ satisfy the conditions in the theorem. From the real Jordan canonical form we have

$$A = S \text{diag}(J_{m_1}(\lambda_1), \ldots, J_{m_q}(\lambda_q), J_{2m_{q+1}}(\lambda_{q+1}), \ldots, J_{2m_p}(\lambda_p))S^{-1},$$

where $S$ is a real $n$-by-$n$ nonsingular matrix, $\lambda_1, \ldots, \lambda_q$ are positive and $\lambda_{q+1}, \ldots, \lambda_p$ are either negative or complex eigenvalues of $A$ that are not necessarily distinct. It is easy to check that the Jordan canonical form of $\log^{(j)} J_{m_k}(\lambda_k)$ is given by

$$\log^{(j)} J_{m_k}(\lambda_k) = \begin{bmatrix} \log^{(j)} \lambda_k & 1/\lambda_k & -1/2\lambda_k^2 & \cdots & (-1)^{m_k-2}\lambda_k^{m_k-2} \\ 0 & \log^{(j)} \lambda_k & 1/\lambda_k & \cdots & (-1)^{m_k-1}\lambda_k^{m_k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/\lambda_k \\ 0 & 0 & 0 & \cdots & \log^{(j)} \lambda_k \end{bmatrix}$$
where \( \log^{(j)} z \) is a branch of \( \log z \) defined by

\[
\log^{(j)} z = \text{Log} z + 2\pi ij, \quad j = 0, \pm 1, \ldots.
\]

As for \( k = q + 1, \ldots, p \), \( J_{2m_k}(\lambda_k) \) has the form

\[
J_{2m_k} = \begin{bmatrix}
L_k & I & 0 & \cdots & 0 \\
0 & L_k & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I \\
0 & 0 & 0 & \cdots & L_k
\end{bmatrix},
\]

where \( L_k = \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix} \) corresponding to either complex conjugate eigenvalues \( \lambda = a_k + ib_k \) and \( \overline{\lambda} = a_k - ib_k \), \( b_k \neq 0 \), each with multiplicity \( m_k \), or to a pair of negative eigenvalues, that is, \( a_k < 0 \) and \( b_k = 0 \). We can use the integration definition to find \( \log^{(j)} J_{2m_k}(\lambda_k) \) for a certain branch of \( \log z \). Namely,

\[
\log^{(j)} J_{2m_k}(\lambda_k) = \frac{1}{2\pi i} \int_{\Gamma} (\log^{(j)} z) (zI - J_{2m_k}(\lambda_k))^{-1} dz,
\]

where \( \Gamma \) encloses the eigenvalues \( \lambda_k, \overline{\lambda_k} \) of \( J_{2m_k}(\lambda_k) \). The inverse \( (zI - J_{2m_k}(\lambda_k))^{-1} \) can be shown to take the general form

\[
(zI - L_k)^{-1} \quad (zI - L_k)^{-2} \quad (zI - L_k)^{-3} \quad \cdots \quad (zI - L_k)^{-m_k} \\
0 \quad (zI - L_k)^{-1} \quad (zI - L_k)^{-2} \quad \cdots \quad (zI - L_k)^{-m_k+1} \\
\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
0 \quad 0 \quad 0 \quad \cdots \quad (zI - L_k)^{-1}
\]

Substituting (6) in (5) and integrating along \( \Gamma \), we have

\[
\log^{(j)} J_{2m_k}(\lambda_k) = \begin{bmatrix}
\log^{(j)} L_k & L_k^{-1} & -\frac{1}{2}(L_k^{-1})^2 & \cdots & \frac{(-1)^{m_k-2}(L_k^{-1})^{m_k-1}}{m_k-2} \\
0 & \log^{(j)} L_k & L_k^{-1} & \cdots & \frac{(-1)^{m_k-3}(L_k^{-1})^{m_k-2}}{m_k-2} \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & L_k^{-1} \\
0 & 0 & 0 & \cdots & \log^{(j)} L_k
\end{bmatrix},
\]

where by using the previous lemma, \( L_k \) has a real logarithm of the form

\[
\log^{(j)} L_k = \begin{bmatrix} \theta_k & \mu_k \\ -\mu_k & \theta_k \end{bmatrix}, \quad \log^{(j)} (a_k \pm ib_k) = \theta_k \pm i\mu_k.
\]
Now set
\[ X = S \text{diag}(\log^{(j_1)}(J_{m_1}(\lambda_1)), \ldots, \log^{(j_q)}(J_{m_q}(\lambda_q)), \log^{(j_{q+1})}(J_{2m_q+1}(\lambda_{q+1})), \ldots, \log^{(j_p)}(J_{2m_p}(\lambda_p)))S^{-1} \]

where for \( k = 1, 2, \ldots, q \), each \( \log^{(j_k)}(J_{m_k}(\lambda_k)) \) is defined by equation (4) and \( \log^{(j_k)}(J_{2m_k}(\lambda_k)) \) is given by (7) for all \( k = q + 1, \ldots, p \). Clearly if we take the logarithms of the Jordan blocks in these forms with particular choice of \( j \) in (4) we can get a real logarithm of \( A \). □

Also, a characterization of the uniqueness of the real logarithm in terms of the spectrum of \( A, \sigma(A) \), is given in the next theorem; Culver [5].

**Theorem 4** Let \( A \in \mathbb{R}^{n \times n} \). Then there exists a unique real logarithm \( X \) of \( A \) if and only if \( A \) is nonderogatory and all the eigenvalues of \( A \) are positive real, that is, if all the eigenvalues of \( A \) are positive and no Jordan blocks of \( A \) belonging to the same eigenvalue appear more than once.

Now we deal with the real polynomial solvability of \( e^X = A \), that is, the existence of polynomial \( p(z) \) such that \( X = p(A) \) and \( e^X = A \). The following theorem establishes the conditions for real logarithm \( X \) of \( A \) to be polynomial in \( A \).

**Theorem 5** Let \( A \in \mathbb{R}^{n \times n} \), be nonsingular matrix with Jordan canonical form
\[ A = S \text{diag}(J_{m_1}(\lambda_1), J_{m_2}(\lambda_2), \ldots, J_{m_p}(\lambda_p))S^{-1} \]

then \( X \in S(A) \) is polynomial in \( A \) if and only if the same value of the scalar logarithm is used for the same eigenvalue of \( A \), that is, if \( e^{x_k} = \lambda_k \) for every \( k = 1, 2, \ldots, p \), then \( \lambda_i = \lambda_j \) implies that \( x_i = x_j \) for all \( 1 \leq i, j \leq p \).

From the previous theorem we conclude that if \( A \in \mathbb{R}^{n \times n} \) has any negative eigenvalues, no real solution of \( e^X = A \) can be polynomial in \( A \).

Suppose that \( A \) is a real \( n \)-by-\( n \) matrix, next we give the additional conditions on \( A \) for which its real logarithm is real normal, symmetric, skew symmetric, positive (semi) positive or orthogonal logarithm. We start by the following lemma on which our results are based.

**Lemma 6** Let \( A \in \mathbb{R}^{n \times n} \) be a nonsingular matrix, and the negative eigenvalues of \( A \), if exist, occur an even number of times, then \( A \) has a real normal logarithm if and only if \( A \) is normal.

**Proof.** Suppose that there exists a real normal logarithm \( X \) of a real matrix \( A \). Then there exists a real orthogonal matrix \( Q \in \mathbb{R}^{n \times n} \) such that
\[ X = Q \text{diag}(D_1, \ldots, D_s, D_{s+1}, \ldots, D_p)Q^T \]
where $D_j$ is 1-by-1 real matrix for all $j = 1, 2, \ldots, s$ and $D_j$ is 2-by-2 real matrix for all $j = s + 1, \ldots, p$, each of them have the form

$$D_j = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$$

corresponding to the complex conjugate eigenvalues $x_j = a_j + ib_j$ and $\bar{x}_j = a_j - ib_j$. Since $e^X = A$, then

$$A = Q\text{diag}(e^{D_1}, \ldots, e^{D_s}, e^{D_{s+1}}, \ldots, e^{D_p})Q^T$$

Clearly, $D_j$ and $D_j^T$ are commuting for all $j = 1, 2, \ldots, p$. It follows that

$$e^{D_j}e^{D_j^T} = e^{D_j + D_j^T}, \quad \text{for all } j = 1, 2, \ldots, p$$

then $AA^T = A^TA$, that is, $A$ is normal.

Conversely, consider that $A$ is normal matrix. Then there exists a real orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$A = Q\text{diag}(D_1, \ldots, D_s, D_{s+1}, \ldots, D_p)Q^T$$

where $D_j$ is a positive number for $j = 1, \ldots, s$, and for $j = s + 1, \ldots p$, $D_j$ is 2-by-2 matrix of the form

$$\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

corresponding to a complex conjugate eigenvalues $a_j + ib_j$ and $a_j - ib_j$, or to a pair of negative eigenvalues, that is, $\alpha < 0$. Then by using Lemma 1 we can construct a real logarithm $X$ of $A$. ■

In the previous lemma, the nonsingular real normal matrix $A$ has a real normal logarithm $X$ with prescribed spectrum $\sigma(X) \subset K$, $K \subset C$ if and only if the scalar equation $e^x = \lambda$ has a solution in $K$ for every $\lambda \in \sigma(A)$.

**Theorem 7** Let $A \in \mathbb{R}^{n \times n}$, be nonsingular real matrix, that has a real logarithm, that is, each Jordan block of $A$ belonging to a negative eigenvalue occurs an even number of times. Then

(a) There exists a symmetric logarithm of $A$ if and only if $A$ is positive definite. This logarithm is unique.

(b) There exists a skew-symmetric logarithm of $A$ if and only if $A$ is orthogonal.

(c) There exists a real positive definite logarithm (positive semidefinite) of $A$ if and only if $A - I > 0$ ($A - I \geq 0$).
(d) There exists an orthogonal logarithm of $A$ if and only if $A$ is normal and for every $\lambda \in \sigma(A)$, we have $|\log \lambda| = 1$, that is, for every $\lambda = a + ib \in \sigma(A)$, we have

$$(\log \sqrt{a^2 + b^2})^2 + (\tan^{-1}\frac{b}{a})^2 = 1$$

**Proof.** (a) The proof of this assertion follows due to the obvious fact that the scalar equation $e^x = \lambda$ has a unique real solution if and only if $\lambda > 0$. Hence the matrix equation $e^X = A$ has a symmetric solution if and only if $A$ is positive definite.

(b) Similarly, the scalar equation $e^x = \lambda$ has a solution belongs to $i\mathbb{R}$ if and only if $|\lambda| = 1$. Then the matrix equation $e^X = A$ has a skew symmetric solution if and only if $A$ is orthogonal.

(c) The logarithm $X$ of $A$ is positive definite (semidefinite) if and only if all the solutions of the scalar equation $e^x = \lambda$ are positive (nonnegative). Namely $X$ is positive definite (semidefinite) if and only if $A$ is a real normal and $\lambda > 1 (\lambda \geq 1)$ for all $\lambda \in \sigma(A)$.

(d) Since all the solutions of the scalar equation $e^x = \lambda$ lie on the unit circle if and only if $|\log \lambda| = 1$, therefore the matrix equation $e^X = A$ has an orthogonal solution if and only if $A$ is real normal and $|\log \lambda| = 1$ for all $\lambda \in \sigma(A)$. ■

### 3 Computation of a real logarithm $X$

In this section we study the problem of computing a real logarithm of a real matrix. Our main tool for such computation is the real Schur decomposition of the real matrix $A$.

Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular real matrix with no negative eigenvalues, then there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$, such that

$$A = QTQ^T = Q \begin{bmatrix} T_{11} & T_{12} & T_{13} & \cdots & T_{1m} \\ 0 & T_{22} & T_{23} & \cdots & T_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T_{mm} \end{bmatrix} Q^T. \quad (8)$$

Here each block $T_{ii}$ is either 1-by-1 or 2-by-2 with complex conjugate eigenvalues $\lambda_i$ and $\overline{\lambda}_i$, $\lambda_i \neq \overline{\lambda}_i$. Since $A$ and $T$ are similar, we have

$$\log A = Q \log T Q^T,$$

so that $\log A$ is real if and only if $\log T$ is real. Hence we need an algorithm for computing a real logarithm of the upper triangular block matrix $T$. If we suppose that $F = \log T = (F_{ij})$, we look for those $F$ which are functions of $T$, 

[Note: The provided text is fragmented and contains mathematical expressions that require proper formatting and notation to be read accurately. The context suggests a discussion on computing the logarithm of a real matrix, focusing on conditions for the existence of an orthogonal logarithm and the computation of such logarithms using the real Schur decomposition.]
and hence $F$ will inherit the upper triangular block structure from $T$. First we compute

$$F_{ii} = \log T_{ii}, \quad \text{for all } i = 1, 2, \ldots, m.$$ 

Once the diagonal blocks of $F$ are known, the blocks in the strict upper triangular of $F$ can be derived from the commutativity result $FT = TF$. Indeed by computing $(i, j)$ entries in this equation, we get

$$\sum_{k=i}^{j} F_{ik} T_{kj} = \sum_{k=i}^{j} T_{ik} F_{kj}, \quad j > i$$

and thus, if $\sigma(T_{ii}) \cap \sigma(T_{jj}) = \emptyset$, $i \neq j$, we obtain an equation with unique solution [2], namely

$$F_{ij} T_{jj} - T_{ii} F_{ij} = T_{ij} F_{jj} - F_{ii} T_{ij} + \sum_{k=i+1}^{j-1} (T_{ik} F_{kj} - F_{ik} T_{kj}), \quad (9)$$

where $F_{ij}$ are computed one superdiagonal at a time. This Sylvester equation results in a linear system of order 1, 2 or 4 that can be solved using standard methods.

From this algorithm for constructing $F$ from its diagonal blocks we conclude that $F$ is real, and consequently $\log A$ is real if and only if each of the blocks $F_{ii}$ is real. Next we discuss the real logarithms $\log T_{ii}$ of 2-by-2 a real matrix with complex conjugate eigenvalues.

**Lemma 8** Let $B = (b_{ij}) \in \mathbb{R}^{2 \times 2}$ with complex conjugate eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, $b \neq 0$. Then $B$ has a countable real logarithms.

**Proof.** Since $B$ has complex conjugate eigenvalues $\lambda$ and $\overline{\lambda}$, $\lambda \neq \overline{\lambda}$, and $\lambda = a + ib$, then there exists a nonsingular matrix $V \in \mathbb{C}^{2 \times 2}$, such that

$$B = V \text{diag}(\lambda, \overline{\lambda}) V^{-1}.$$ 

Then $B$ can be written in the form $B = aI + ibKV^{-1} = aI + bW$, where $W = iVKV^{-1}$ and $K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Clearly $W$ is a real matrix. Thus any logarithm $X$ of $B$ is given by

$$X = V \text{diag}((\log(j_1) \lambda, \log(j_2) \overline{\lambda})) V^{-1}$$

$$= V \text{diag}(\theta + i\mu + i2\pi j_1, \theta - i\mu + i2\pi j_2) V^{-1}$$

where $\theta = \log |\lambda|$, $\mu = \text{Arg}\lambda$ and $\text{Arg}\overline{\lambda} = -\text{Arg}\lambda$. Then the set of all logarithms of a 2-by-2 real matrix (with complex conjugate eigenvalues) is a countable set, each logarithm is given by

$$X = \theta I + \mu W + VE V^{-1} \quad (10)$$
where $E = i2\pi \text{diag}(j_1, j_2)$. In fact equation (10) gives all the possible solutions of $e^X = B$. The logarithm in equation (10) is real if and only if $VEV^{-1}$ is a real matrix, that is, if and only if $j_1 = -j_2$. In this case $VEV^{-1} = i2\pi j_1 VKV^{-1} = 2\pi j_1 W$. Then any real logarithm $X$ of $B$ has the form

$$X = \theta I + (\mu + 2\pi j_1)W,$$

where $W = \frac{1}{i}(B - aI)$. And indeed once $\theta$ and $\mu$ are known we have a countable set of real logarithms. $\blacksquare$

The set of real logarithms of a real 2-by-2 matrix (with complex conjugate eigenvalues) can also obtained in an alternative approach by using Lagrange interpolation as follows.

Let $B = (b_{ij}) \in R^{2 \times 2}$ with complex conjugate eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, then there exists a polynomial $r(z)$ of the first degree given by

$$r(z) = (\log^{(j_1)} \lambda) \frac{(z - \overline{\lambda})}{(\lambda - \overline{\lambda})} + (\log^{(j_2)} \overline{\lambda}) \frac{(z - \lambda)}{(\overline{\lambda} - \lambda)}$$

where $\log^{(j)} z$ is a branch of $\log z$. Hence we can define a logarithm $X$ of $B$ as

$$X = r(B) = (\log^{(j_1)} \lambda) \frac{(B - \overline{\lambda}I)}{2ib} + (\log^{(j_2)} \overline{\lambda}) \frac{(B - \lambda I)}{-2ib}$$

$$= \frac{i}{2b} \left[(\log^{(j_2)} \overline{\lambda})(B - \lambda I) - (\log^{(j_1)} \lambda)(B - \overline{\lambda}I)\right]$$

$$= \frac{i}{2b} \left[(\log \overline{\lambda} + i2\pi j_2)(B - \lambda I) - (\log \lambda + i2\pi j_1)(B - \overline{\lambda}I)\right]$$

$$= \frac{i}{2b} \left[(\log \lambda - i2\pi j_2)(B - \overline{\lambda}I) - (\log \lambda + i2\pi j_1)(B - \overline{\lambda}I)\right]$$

where $j_1, j_2 \in Z$. This logarithm is real if the matrix in the bracket is pure imaginary, that is, if $j_1 = -j_2$. For example if we set $j_1 = j_2 = 0$, we have the principal logarithm $\text{Log} B$, namely

$$\text{Log} B = \frac{i}{2b} \left[(\log \lambda)(B - \overline{\lambda}I) - (\log \lambda)(B - \overline{\lambda}I)\right]$$

$$= \frac{i}{2b} 2i \text{Im} [(\log \lambda)(B - \overline{\lambda}I)]$$

$$= \frac{1}{b} \text{Im} [(\log \lambda)(B - \overline{\lambda}I)].$$

Hence

$$\text{Log} B = \frac{1}{b} \left[\begin{array}{cc}
b \log |\lambda| + (b_{11} - a) \text{Arg} \lambda & b_{12} \text{Arg} \lambda \\
b_{21} \text{Arg} \lambda & b \log |\lambda| + (b_{22} - a) \text{Arg} \lambda\end{array}\right], \quad (11)$$
where $a = \frac{1}{2}(b_{11} + b_{22})$ and $b = \frac{1}{2}\sqrt{-(b_{11} - b_{22})^2 - 4b_{12}b_{21}}$.
We summarize the previous steps in the following algorithm.

**Algorithm** special-real-logarithm $(B, j_1)$
(This algorithm computes a real logarithm of a real 2-by-2 matrix $B$.)

**Input:** $A, j_1$

$a = (b_{11} + b_{22})/2$;

$b = \sqrt{-((b_{11} - b_{22})^2 - 4b_{12}b_{21})}/2$;

$\theta = \frac{1}{2}\log(a^2 + b^2)$;

$\mu = \tan^{-1}(b/a)$;

$X = \theta I + \frac{1}{4}(\mu + 2\pi j_1)(B - aI)$.

Now we can give an algorithm to compute a real logarithm of a block upper triangular real matrix $T$. Assume that $T$ defined by equation (8) such that $T_{11}, \ldots , T_{rr}$ are 1-by-1 and $T_{r+1,r+1}, \ldots , T_{mm}$ are 2-by-2 matrices with complex conjugate eigenvalues. The following algorithm compute a real logarithm $F$ of $T$.

**Algorithm** general-real-logarithm

**Input:** $T$

for $i = 1$ to $r$

$F_{ii} = \log(T_{ii})$

endfor

for $i = r + 1$ to $m$

$F_{ii} = \text{special-real-logarithm} \ (B, j_1)$

endfor

$SUM = 0$;

for $i = 1$ to $m$

for $j = 2$ to $m$

if $(j - 1 \geq i + 1)$ then

for $k = i + 1$ to $j - 1$

$SUM = SUM + T_{ik}F_{kj} - F_{ik}T_{kj}$

endfor

endif

Solve $F_{ij}T_{jj} - T_{ii}F_{ij} = T_{ij}F_{jj} - F_{ij}T_{ij} + SUM$.
(This system of equations can be solved by any standard method)

endfor

endfor.

Note that, if $A$ is a real normal matrix then the above algorithm computes the real logarithms even if $A$ has negative or repeated eigenvalues provided that the negative eigenvalues occur in pairs.
3.1 Real logarithm of real normal matrix

If $A \in \mathbb{R}^{n \times n}$ is a normal matrix and each of its negative eigenvalue occurs an even number of times, then Theorem 6 implies that there exists a real orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$A = Q \text{diag}(D_1, \ldots, D_s, D_{s+1}, \ldots, D_p)Q^T,$$

where $D_j$ is a positive number for $j = 1, \ldots, s$, and for $j = s+1, \ldots p$, $D_j$ is a 2-by-2 matrix of the form

$$\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix} \text{ or } \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

corresponding to a pair of complex conjugate eigenvalues $a_j + ib_j$ and $a_j - ib_j$, or to a pair of negative eigenvalues, that is, $\alpha < 0$. Then by using Lemma 1 we can find a real matrix $X_j$ such that $e^{X_j} = D_j$ for all $j = 1, 2, \ldots, p$, and consequently

$$X = Q \text{diag}(X_1, X_2, \ldots, X_j)Q^T$$

is a real logarithm of $A$. If $A$ has negative eigenvalues then there is no real logarithm of $A$ which is a polynomial in $A$.

Example 9 Consider the normal matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

with eigenvalues $6, -1.5 \pm 0.866i$. The real Schur decomposition of $A$ is given by $A = QTQ^T = Q \text{diag}(D_1, D_2)Q^T$, where

$$Q = \begin{bmatrix} 0.5774 & 0.3004 & 0.7592 \\ 0.5774 & 0.5073 & -0.6397 \\ 0.5774 & -0.8077 & -0.1195 \end{bmatrix},$$

and

$$T = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -1.5 & 0.866 \\ 0 & -0.866 & -1.5 \end{bmatrix}.$$  

Then by using Lemma 1, there exists a real logarithm $X$ of $A$ of the form

$$X = Q \text{diag}(X_1, X_2)Q^T,$$

where $X_1 = 1.792$ and $X_2 = \begin{bmatrix} 0.549 & 2.618 \\ -2.618 & 0.549 \end{bmatrix}$. Consequently

$$X = \begin{bmatrix} 0.9634 & -1.0969 & 1.9258 \\ 1.9259 & 0.9634 & -1.0969 \\ -1.0970 & 1.9258 & 0.9634 \end{bmatrix}.$$
References


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